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Bifurcations in nonlinear models of fluid-conveying pipes supported at both ends

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Abstract

Stationary bifurcations in several nonlinear models of fluid conveying pipes fixed at both ends are analyzed with the use of Lyapunov–Schmidt reduction and singularity theory. Influence of the gravitational force, curvature and vertical elastic support on various properties of bifurcating solutions are investigated. In particular the conditions for occurrence of supercritical and subcritical bifurcations are presented for the models of Holmes, Thurman and Mote, and Paidoussis.

Keywords: Bifurcations; Fluid conveying pipes; Nonlinear models

1. Introduction

The nonlinear dynamics behavior of pipes conveying fluid displays interesting and paradigmatic behavior with important practical implications, and it offers a rich setting for development and testing of nonlinear dynamics theory. Bifurcation theory represents one of the main subject areas of nonlinear dynamics, analyzing behavior either in the vicinity of trivial solutions (local bifurcations), or the existence in-the-large of connected sets of nontrivial solutions (global bifurcations). In the study of local bifurcations, particularly interesting is the influence of various parameters figuring in the governing equations on the location and stability of fixed points and on classification of bifurcations as supercritical or subcritical. In a supercritical bifurcation there is no discontinuous change in size and form of the attractor, and after the bifurcation the new (enlarged) attractor contains within itself the old attractor. On the other hand, in a subcritical bifurcation the attractor disappears followed by a jump of the system to a remote and completely new attractor, via a fast dynamic transient. For the complete understanding of the dynamics, both stationary and nonstationary (dynamic) aspects of the bifurcation theory are important.

In the case of fluid-conveying pipes, nonstationary analysis is related to the study of supercritical and subcritical Hopf bifurcations, the determination of the amplitude associated with flutter and the dependence of oscillation frequency on the amplitude. Stationary analysis, on the other hand, is concerned with stationary bifurcations i.e. changes in the equilibrium point structure of the underlying equations, which due to reflection symmetry are of pitchfork type. Furthermore interplay between stationary and dynamic bifurcations may lead to more complicated dynamics such as spatio-temporal intermittency and chaos (Argentina and Coullet, 1998). In spite of many publications devoted

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to these subjects, a thorough and systematic analysis of nonlinear models of fluid conveying pipes, particularly of pipes supported at both ends, and related bifurcations is still lacking. Following the pioneering work by Holmes (1977, 1978) further work of interest appeared in Ch'ng (1977, 1979) and Lunn (1982). As a paradigm of the studies performed so far, the work of Holmes based on the center manifold analysis concentrated on a one-equation model which does not take into account any gravitational nor tensioning effects. The coefficients figuring in the normal form of the obtained bifurcation equation were given only as numerical values, disguising the influence of involved physical quantities. Finally, the complete two-equation model of Païdoussis (2003) has only recently been introduced in its correct form and it has never been used. A wealth of up-to-date information related to the published work on nonlinear aspects of fluid-conveying pipes with supported ends may be found in Païdoussis (1998, 2003).

In order to improve upon the current status and in order to prepare the ground for the study of spatio-temporal intermittency (and chaos), we study several well-known and frequently used nonlinear models in which we focus on nondegenerate local bifurcations from the trivial solution to another stationary nontrivial solution. In particular, the object of our analysis are pipes fixed at both ends shown in Fig. 1 and the nonlinear models of Holmes (1977), Thurman and Mote (1969) and the complete nonlinear model of Païdoussis (2003) [the latter being based on the model of Semler et al. (1994)]. The manner of presentation is such that each model is considered as a special case of the complete nonlinear model, revealing the influence on the complete dynamics of different linear and nonlinear terms and quantities. Moreover the increasing complexity of each model leads to a better understanding of the complete nonlinear model. Each model is presented in a separate section of the paper, with subsections related to certain important parameters influencing the dynamics, such as the gravitational force or pipe curvature.

Since this exposition relies strongly on the Lyapunov–Schmidt reduction, a brief description of the procedure is presented in Appendix A. The interested reader may find extensive treatment in books (Golubitsky et al., 1985; Golubitsky and Schaeffer, 1988). An important characteristic of the Lyapunov–Schmidt reduction is that the procedure for obtaining normal form of the bifurcation equation is pursued by analytical means, without using any numerical or truncating procedures. Hence the normal form is exact and the influence of each parameter on the type of bifurcation may be traced and analyzed. An important insight gained by using this method is reflected in the fact that we obtain exact analytical solutions in the vicinity of a bifurcation point for each model and derive exact conditions that classify bifurcations as supercritical or subcritical.

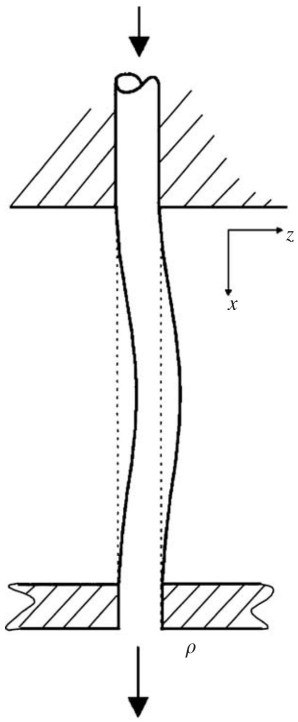

Fig. 1. A pipe with supported ends conveying fluid. The diagram also shows coordinates used in the text; ρ represents the fluid velocity.

2. The model of P.J. Holmes

2.1. Bifurcations in the stationary model without gravitational effects

Holmes considered pipes with supported, nonsliding ends and obtained a nonlinear equation of motion by adding to the linear equation a nonlinear term corresponding to the mean, deformation-induced tensioning. The complete equation of motion in nondimensional form reads

$$\alpha \dot{v}'''' + v'''' - (\Gamma - \rho^2 - \tfrac{1}{2}\mathscr{A}|v'|^2 + \alpha\mathscr{A}\langle v'|\dot{v}'\rangle)v'' - \gamma([1-\xi]v')' + 2\sqrt{\beta}\rho\dot{v}' + \sigma\dot{v} + \ddot{v} = 0, \tag{1}$$

where $v = v(\xi, t)$ denotes the lateral deflection normalized by the length of the pipe. In addition, Γ represents the tensile force on the pipe, β represents the mass ratio, ρ the flow velocity, $\mathscr{A}$ the axial stiffness, α is related to the viscoelastic structural damping, σ represents fluid damping, and γ denotes gravitational effects. Explicitly,

$$\alpha = \left(\frac{EI}{M+m}\right)^{1/2} \frac{a}{L^2}, \quad \beta = \frac{M}{M+m}, \quad \gamma = \frac{M+m}{EI} L^3 g, \tag{2}$$

where a is the coefficient of Kelvin–Voigt damping in the pipe material, M and m are the mass of fluid and pipe per unit length, respectively, and L is the length of the pipe, while

$$\Gamma = \frac{T_0 L^2}{EI}, \quad \mathscr{A} = \frac{EAL^2}{EI}, \quad \sigma = \frac{cL^2}{[EI(M+m)]^{1/2}}. \tag{3}$$

In the above equations T_0, EI, L, A, I, E and c are the longitudinal externally applied tension, flexural rigidity of the pipe, length of the pipe, cross-sectional area, area-moment of inertia, modulus of elasticity and constant of damping (due to friction), respectively. Moreover $v' = \mathrm{d}v/\mathrm{d}\xi$, $\dot{v} = \mathrm{d}v/\mathrm{d}t$. With $\xi = s/L$, $|v|$ denotes the L^2 norm

$$|v'| = \left[\int_0^1 v'^2(\xi)\,\mathrm{d}\xi\right]^{1/2}, \tag{4}$$

while the $\langle v'|\dot{v}'\rangle$ denotes the L^2 inner product

$$\langle v'|\dot{v}'\rangle = \int_0^1 v'(\xi)\dot{v}'(\xi)\,\mathrm{d}\xi. \tag{5}$$

The boundary conditions are $v = v'' = 0$ at $\xi = 0, 1$. The corresponding stationary equation, neglecting gravitational effects, applied tension and nonlinear dissipative term following a Lyapunov–Schmidt reduction, locally may be put into one-to-one correspondence with solutions of the single algebraic equation

$$g(\mu, \rho) = \left(\tfrac{3}{4}(n\pi)^4\mathscr{A}\right)\mu^3 - (n\pi)^3\mu\rho + \vartheta(3) = 0, \tag{6}$$

indicating that the corresponding bifurcation is always a supercritical pitch-fork one, since

$$\tfrac{3}{4}(n\pi)^4\mathscr{A} > 0.$$

Using a two-mode Galerkin discretization applied to the complete equation (1) and assuming $\sigma = g = \Gamma = 0$, Holmes (1977) obtained numerical values for terms multiplying $\mathscr{A}\mu^3$ and $\mu\rho$, respectively, in the normal form equation (6). We have shown here that the sign of these terms is unalterable and we have presented their analytical form. Applying the Lyapunov–Schmidt reduction to the same stationary equation and taking applied tension into account, the following normal form of the bifurcation equation is obtained:

$$g(\mu, \rho) = \frac{3}{4}\left(n\pi + \frac{\Gamma}{2n\pi}\right)^4 \mathscr{A}\left[1 - \frac{\sin\left(\dfrac{\Gamma}{n\pi}\right)}{4\left(n\pi + \dfrac{\Gamma}{2n\pi}\right)^2}\right]\mu^3 - (n\pi)^3\mu\rho + \vartheta(3) = 0. \tag{7}$$

In this case too, the type of pitchfork bifurcation is supercritical and may not be altered, irrespective of the value of tension Γ or axial flexibility $\mathscr{A}$.

Since the influence of gravitational effects was not considered in the original treatment (Holmes, 1977, 1978), nor in subsequent studies, we analyze this effect along with the effect of tensioning in the next section, again restricting analysis to the stationary version of Eq. (1).

2.2. The role of gravitational effects

The stationary equation representing nonlinear dynamic behavior of a pipe conveying fluid including gravitational and tensioning effects may be written in the form:

$$v'''' - \left(\rho^2 - \Gamma - \tfrac{1}{2}\mathscr{A}|v'|^2\right)v'' - \gamma([1-\xi]v')' = 0, \tag{8}$$

where v, as in (1) denotes the displacement in the z-direction. The boundary conditions are $v = v'' = 0$ at $\xi = 0$ and $\xi = 1$. Introducing the variable $\alpha^2 = \rho^2 - (\Gamma + \gamma)$, integrating once and setting $\mathrm{d}v/\mathrm{d}\xi = u$ we obtain the equation

$$u'' + (\alpha^2 + \gamma\xi)u - \tfrac{1}{2}\mathscr{A}|u'|^2 = 0, \tag{9}$$

with boundary conditions $u'(\xi = 0) = u'(\xi = 1) = 0$. The one-dimensional kernel of the corresponding linear operator

$$L{:=}u''(\alpha^2 + \gamma\xi)u,$$

is now spanned not by a sine function, as in the case without gravitational effects, but by Bessel functions. The proof of this proposition requires transformation of the equation

$$u'' + (\alpha^2 + \gamma\xi)u = 0, \tag{10}$$

to the equation

$$t^2w'' + tw' + (t^2 - (\tfrac{1}{3})^2)w = 0, \tag{11}$$

where in the above equation the prime denotes differentiation with respect to t. First, substitution $(\alpha^2 + \gamma\xi)\gamma^\sigma = y$, $\sigma = -\frac{2}{3}$, leads to the equation

$$u'' + yu = 0,$$

where the prime denotes differentiation with respect to y. The subsequent substitution $t = \frac{2}{3}y^{3/2}$, $u = \sqrt{y}w$ leads to Eq. (11). This equation may be solved in terms of Bessel functions so that the corresponding solution of the linearized equation (8) is

$$v(\xi) = (\alpha^2 + \gamma\xi)\left[\mathrm{J}_{2/3}\left(\frac{2}{3}\frac{\alpha^3}{\gamma}\right)\mathrm{J}_{-2/3}\left(\frac{2}{3}\frac{(\alpha^2+\gamma\xi)^{3/2}}{\gamma}\right) - \mathrm{J}_{-2/3}\left(\frac{2}{3}\frac{\alpha^3}{\gamma}\right)\mathrm{J}_{2/3}\left(\frac{2}{3}\frac{(\alpha^2+\gamma\xi)^{3/2}}{\gamma}\right)\right]. \tag{12}$$

For gravity and fluid motion in the same direction, and assuming $\Gamma = 0$, the location of bifurcation points (nontrivial equilibria) is obtained from the expression

$$\rho_n = n\pi + \frac{1}{4}\frac{\gamma}{n\pi} + \vartheta(\gamma^2), \quad n = 0, 1, 2, \ldots\ . \tag{13}$$

Details of the derivation of Eqs. (10) and (13) are presented in Appendix B. The above expression suggests that bifurcation points are shifted as compared to the case without gravity in the direction of increasing velocity by the amount $\gamma/4n\pi$. Correspondingly, bifurcations occur for higher velocities compared to the case when gravity is not taken into account. As the velocity increases, the effects of gravity are weaker, so that finally locations of bifurcation points are the same as for the case without gravity. This is an intuitively pleasing result as one expects high enough velocity to annul the effects of gravity. If Γ is not neglected, the expression for the critical velocity is

$$\begin{aligned}\rho_n &= n\pi + \frac{1}{4}\frac{\gamma}{n\pi} + \frac{1}{2}\frac{\Gamma}{n\pi} + \vartheta(2),\\ &= n\pi + \frac{1}{4}\frac{\gamma + 2\Gamma}{n\pi} + \vartheta(2), \quad n = 1, 2, \ldots\end{aligned} \tag{14}$$

The resulting bifurcation diagrams are presented in Fig. 2. Hence the roles of tensioning and gravity are similar; however, as the velocity increases, the influence of gravity diminishes twice as fast as tensioning effects. The other aspect of this phenomenon may be observed by considering the distance between bifurcation points, in units of velocity, given by the following expression:

$$\Delta\rho_n = \pi - \frac{1}{4\pi}\frac{\gamma}{n(n+1)} - \frac{1}{2\pi}\frac{\Gamma}{n(n+1)}. \tag{15}$$

When $\gamma = \Gamma = 0$, bifurcation points are equidistant from each other, this distance being equal to π. When γ and Γ are not equal to zero, the distance is an increasing sequence, whose limit ($n \to \infty$) is equal to π.

If the direction of the fluid is opposite to the direction of gravity and assuming no tensioning effects, bifurcation points are located at

$$\rho_n = n\pi - \frac{1}{4}\frac{\gamma}{n\pi} + \vartheta(\gamma^2), \quad n = 0, 1, 2, \ldots \ . \tag{16}$$

In this case, bifurcations occur for velocities smaller than in the case without gravity, and again for high enough velocities the effects of gravity on the location of bifurcation points are annulled. Inclusion of tensioning effects yields

$$\rho_n = n\pi - \frac{1}{4}\frac{\gamma}{n\pi} + \frac{1}{2}\frac{\Gamma}{n\pi} + \vartheta(2) = n\pi - \frac{1}{4}\frac{\gamma - 2\Gamma}{n\pi} + \vartheta(2), \quad n = 1, 2, \ldots,$$

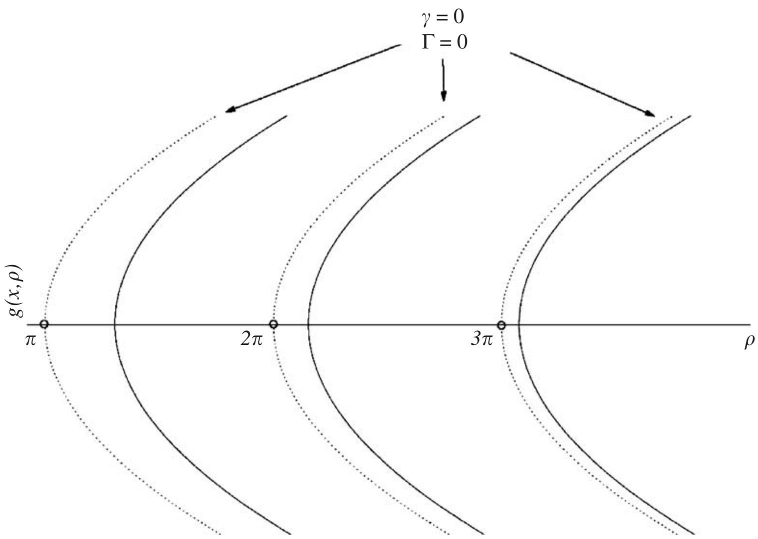

Fig. 2. Supercritical pitchfork bifurcation for the model of Holmes with and without gravity and tensioning effects. Fluid row is in the same direction as gravity.

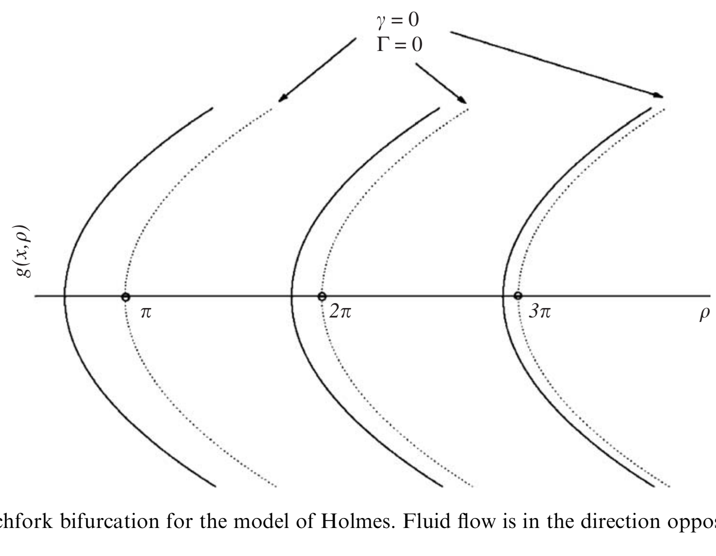

Fig. 3. Supercritical pitchfork bifurcation for the model of Holmes. Fluid flow is in the direction opposite to the direction of gravity.

and the corresponding bifurcation diagrams are presented in Fig. 3. The matching bifurcation interpoint distances are

$$\Delta\rho_n = \pi + \frac{1}{4}\frac{\gamma}{n(n+1)} - \frac{1}{2\pi}\frac{\Gamma}{n(n+1)}.$$

In this case, there is a competition between gravity and tensioning effects, with the possibility that they annul each other when $\gamma = 2\Gamma$.

Eq. (11) yields an approximate solution

$$v(\xi) \simeq \frac{1}{(\alpha^2+\gamma\xi)^2}\sin\left[\frac{2}{3}\frac{(\alpha^2+\gamma\xi)^{3/2}-\alpha^3}{\gamma}\right], \tag{17}$$

which for $\gamma \to 0$ approaches the solution $\sin(n\pi\xi)$, the solution of the linear equation corresponding to (8) without gravitational effects. In order to shed more light on the physical aspect of the above results, it is instructive to call upon an important relationship between the solution $\mu(\rho)$ of the algebraic equation $g(\mu, \rho) = 0$ and the solution of the full problem $v(\xi)$. Near the bifurcation point, the nontrivial solutions of (8) have the spatial structure of the basis vector u_1 that spans the one-dimensional kernel of the corresponding linear operator. Hence, the solutions μ of the reduced bifurcation equation $g(\mu, \rho) = 0$ are related to solutions of Eq. (8) as

$$v = \mu u_1 + \vartheta(\mu^2),$$

so that the complete solution of (8) in the vicinity of the bifurcation point may be written as

$$v(\xi) = \mu(\rho)(\alpha^2+\gamma\xi)\left[\mathrm{J}_{2/3}\left(\frac{2}{3}\frac{\alpha^3}{\gamma}\right)\mathrm{J}_{-2/3}\left(\frac{2}{3}\frac{(\alpha^2+\gamma\xi)^{3/2}}{\gamma}\right) - \mathrm{J}_{-2/3}\left(\frac{2}{3}\frac{\alpha^3}{\gamma}\right)\mathrm{J}_{2/3}\left(\frac{2}{3}\frac{(\alpha^2+\gamma\xi)^{3/2}}{\gamma}\right) + \vartheta(\xi^2)\right], \tag{18}$$

satisfying boundary conditions $v(0) = v(1) = 0$; $v''(0) = v''(1) = 0$.

The analysis of the Holmes model reveals several important features. Irrespective of the inclusion or exclusion of gravity and/or tensioning effects, the stationary bifurcation is of supercritical type. Although gravity and tension have a similar effect on the location of bifurcation points, only gravity changes the solution locally. In the vicinity of the bifurcation point, the solution (eigenfunction) is a sine function for $\gamma = 0$ and a Bessel function for $\gamma \neq 0$. Moreover, both gravity and tension increase the distance between bifurcation points along the parameter space and this effect is particularly notable for low velocities, while high velocities compensate the influence of gravity and tension as intuitively expected.

3. The model of Thurman and Mote

3.1. Bifurcations in the stationary model

This model (Thurman and Mote, 1969; Païdoussis, 1998), which considers both lateral and axial deflections, was derived under the following assumptions: (i) no gravity force, (ii) steady flow velocity, (iii) linear moment–curvature relationship, and (iv) a simple approximation of the fluid velocity. For the study of the normal form of the bifurcation equations it is more instructive to consider this model as a special case of the complete nonlinear model of Païdoussis (2003),[1] to be analyzed in the next section. The nondimensional equations for spatial motion of the complete nonlinear model are

$$\begin{aligned} &w'''' + (\rho^2 - (\Gamma - \Pi))w'' + \gamma w' + (\Gamma - \mathscr{A} - \Pi)(w''u' + u'w'' + \tfrac{3}{2}w'^2w'') \\ &\quad - (3u'''w'' + 4u''w''' + 2u'w'''' + w'u'''' + 2w''^3 + 2w'^2w'''' + 8w'w''w''') \\ &\quad - \gamma[w'u' + \tfrac{1}{2}w'^3 - (1-\xi)(-w'' + u''w' + u'w'' + \tfrac{3}{3}w'^2w'')] = 0, \end{aligned} \tag{19}$$

$$(\rho^2 - \mathscr{A})u'' - (w''w''' + w'w'''') - \gamma[\tfrac{1}{2}w'^2 - (1-\xi)w'w''] + (\Gamma - \mathscr{A} - \Pi)w'w'' = 0, \tag{20}$$

where the dimensionless tension and the pressure at the downstream end are

$$\Gamma = \frac{T(L)L^2}{EI}, \quad \Pi = \frac{P(L)L^2}{EI}, \tag{21}$$

[1] The corrected version of equations is given in Appendix T.4.

respectively, while u denotes the dimensionless longitudinal deflection in the x direction (direction of gravity), and the other quantities being the same as in the model of Holmes, with w here replacing v. A prime denotes differentiation with respect to the Lagrangian variable x which may be used interchangeably with ξ. The following assumptions may be attributed to the model of Thurman and Mote:

$$\gamma = 0, \quad \Pi = 0, \quad \Gamma = \frac{T_0 L^2}{EI}, \quad \kappa^2 = 0,$$

where $T_0 = \text{const.}$, denotes externally applied tension and κ^2 denotes curvature.

Consequently the equations of motion of this model are

$$w'''' + (\rho^2 - \Gamma)w'' + (\Gamma - \mathscr{A})\big(w''u' + u'w'' + \tfrac{3}{2}w'^2 w''\big) = 0, \quad (\rho^2 - \mathscr{A})u'' + (\Gamma - \mathscr{A})w'w'' = 0. \tag{22}$$

The boundary conditions are $u(0) = w(0) = u(\xi = 1) = w(\xi = 1) = 0$, with the additional condition $w''(0) = w''(\xi = 1) = 0$ for a simply supported pipe or $w'(0) = w'(\xi = 1) = 0$ for a clamped–clamped one.

Fixed points, as in the model of Holmes for $\gamma = 0$, are located at positions

$$\rho_n = n\pi + \frac{1}{2}\frac{\Gamma}{n\pi} + \vartheta(\Gamma^2), \quad n = 0, 1, 2, \ldots .$$

as easily seen from the following argument. The kernel of the linear operator L corresponding to the system consisting of Eqs. (19) and (20) is obtained by solving the equation

$$L\begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} w_0'''' + (\rho^2 - \Gamma)w_0'' \\ u_0'' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{23}$$

where

$$\begin{pmatrix} w_0(x) \\ u_0(x) \end{pmatrix} = \sum_{n=1}^{\infty} \begin{pmatrix} a_n \\ b_n \end{pmatrix} \sin n\pi\xi, \tag{24}$$

satisfying boundary conditions

$$\begin{pmatrix} w_0(0) \\ u_0(0) \end{pmatrix} = \begin{pmatrix} w_0(1) \\ u_0(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence, the position of bifurcation points is readily obtained from Eqs. (23) and (24). A straightforward calculation indicates that the kernel of L is spanned by

$$\operatorname{Ker} L\left(\left(\begin{pmatrix} w \\ u \end{pmatrix}, \rho\right)\right) = \mathbb{R}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin n\pi\xi\right\}. \tag{25}$$

Calculation of terms defined in Appendix A, leads to the following bifurcation equation

$$g(\mu, \rho) = \tfrac{3}{8}(n\pi)^4[\alpha(\beta - 3)]\mu^3 - (n\pi)^3\mu\rho + \vartheta(x^3) = 0, \tag{26}$$

where

$$\alpha = \Gamma - \mathscr{A}, \quad \beta = \frac{\Gamma - \mathscr{A}}{\rho^2 - \mathscr{A}}.$$

In the vicinity of the bifurcating solutions, the solutions of the complete model have the form

$$\begin{pmatrix} w(\xi) \\ u(\xi) \end{pmatrix} = \mu(\rho)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin k\pi\xi + \vartheta(\mu^2), \tag{27}$$

where $\mu(\rho)$ represents the solution of Eq. (26). Hence, again the bifurcation is of the pitchfork type; however, there are important differences between the model of Holmes and the model of Thurman and Mote. It is immediately evident that the bifurcation may be of supercritical pitchfork type for

$$\alpha(\beta - 3) > 0,$$

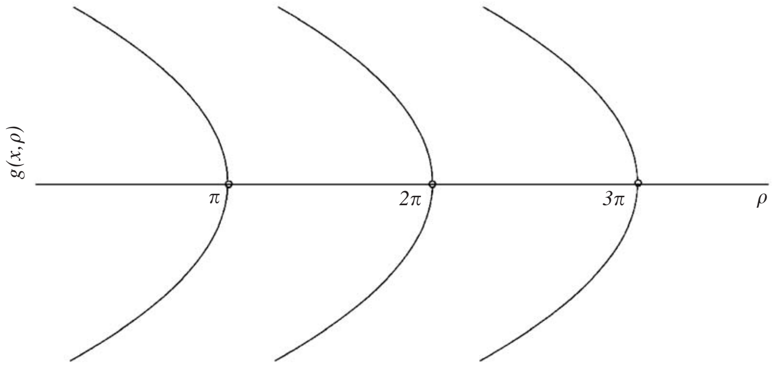

Fig. 4. Subcritical pitchfork bifurcation for the model of Thurman and Mote.

and of subcritical type, shown in Fig. 4, for

$$\alpha(\beta - 3) < 0.$$

Writing explicitly the expression on the left side of these inequalities as

$$\frac{(\Gamma - \mathscr{A})^2}{\rho_n^2 - \mathscr{A}} - 3(\Gamma - \mathscr{A}),$$

and recalling that $\rho_n^2 = \Gamma + (n\pi)^2$, implies that two cases may be analyzed, depending on whether $|\alpha|/\rho_n^2 < 1$ or $\rho_n^2/|\alpha| < 1$. However, since only the first point of instability corresponding to $n = 1$ is relevant from the physical point of view, the former condition is of no practical importance although it is important for understanding the mathematical aspects of the model.

Case 1: $|\alpha|/\rho_n^2 < 1$. This situation corresponds to high fluid velocities (large n). In this case, a straightforward calculation yields the following conditions for the occurrence of supercritical and subcritical bifurcations:

Supercritical condition:

$$\Gamma < \mathscr{A}. \tag{28}$$

Subcritical condition:

$$\Gamma > \mathscr{A}. \tag{29}$$

Based on (3) the above inequalities may be also expressed as

$$T_0 < EA,$$

and

$$T_0 > EA.$$

Case 2: $\rho_n^2/|\alpha| < 1$. This case is applicable to low fluid velocities (small n) and large $|\Gamma - \mathscr{A}|$, so that either a highly flexible pipe is considered or the effects of tensioning are large. Conditions for the occurrence of supercritical and subcritical bifurcations are

Supercritical case:

$$\Gamma < \mathscr{A} - \tfrac{1}{2}(n\pi)^2. \tag{30}$$

Subcritical case:

$$\Gamma > \mathscr{A} - \tfrac{1}{2}(n\pi)^2. \tag{31}$$

For low fluid velocities, both conditions strongly depend on values of Γ and $\mathscr{A}$, since n is small, so that supercritical bifurcation is more likely in a pipe of high axial flexibility and low externally applied tension while subcritical bifurcation is more likely in a pipe with small axial flexibility and high externally applied tension.

Hence, an important feature of the Thurman and Mote model is dependence of the bifurcation type (supercritical or subcritical) on the relationship between elastic characteristics of the pipe and externally applied tension, with the velocity-dependent term figuring only in inequalities relevant at low fluid velocities (the case of physical validity). At this point it should be mentioned that if ρ^2 does not appear[2] in the equation for u, only one classification condition is obtained in the form of inequalities (28) and (29); thus, from the point of view of the bifurcation theory, this version of the model assumes high fluid velocities. On the other hand, since the case of high fluid velocities requires large n and is therefore of no practical importance, the term ρ^2 is necessary for the model to be useful in bifurcation analysis. Comparison with the model of Holmes, which assumes just lateral deflections and for which only super-critical bifurcation is possible, reveals that inclusion of the equation for axial deflections makes subcritical bifurcation also possible.

3.2. Thurman and Mote model with curvature

The inclusion of the curvature term $EI\kappa^2$ in the model of Thurman and Mote may again be considered as a special, reduced version of the complete nonlinear model. Retaining simplifications of the Thurman and Mote model in effect, namely $\gamma = 0$, $\Pi = 0$, $\Gamma = T_0L^2/EI$, $\kappa^2 = 0$, $T_0 = \text{const.}$, but keeping terms that arise from the curvature effect, the following dimensionless equations of motion are obtained:

$$w'''' + (\rho^2 - \Gamma)w'' + (\Gamma - \mathscr{A})(w''u' + u'w'' + \tfrac{3}{2}w'^2w'') \\ - (3u'''w'' + 4u''w''' + 2u'w'''' + w'u'''' + 2w''^3 + 2w'^2w'''' + 8w'w''w''') = 0, \tag{32}$$

$$(\rho^2 - \mathscr{A})u'' - (w''w''' + w'w'''') + (\Gamma - \mathscr{A})w'w'' = 0. \tag{33}$$

Boundary conditions are the same as in the basic Thurman and Mote model. Arguments used in previously discussed models lead, as expected, to the same locations of fixed points:

$$\rho_n = n\pi + \frac{1}{2}\frac{\Gamma}{n\pi} + \vartheta(\Gamma^2), \quad n = 0, 1, 2, \ldots\ .$$

Evaluation of terms of the bifurcation equation shows that it has the following form:

$$g(x, \rho) = \tfrac{3}{4}(n\pi)^4[(7 - \beta)(n\pi)^2 - \tfrac{1}{2}\alpha(\beta + 3)]x^3 - (n\pi)^3 x\rho, \tag{34}$$

where

$$\alpha = \Gamma - \mathscr{A}, \quad \beta = \frac{\Gamma - \mathscr{A} + 2(n\pi)^2}{\rho_n^2 - \mathscr{A}} = 1 + \frac{(n\pi)^2}{\alpha + (n\pi)^2}.$$

Hence, the bifurcation is of supercritical type if

$$(7 - \beta)(n\pi)^2 - \tfrac{1}{2}\alpha(\beta + 3) > 0,$$

and subcritical if

$$(7 - \beta)(n\pi)^2 - \tfrac{1}{2}\alpha(\beta + 3) < 0.$$

More insight into these inequalities is gained by considering specific cases dependent on the fluid velocity.

Case 1: $|\alpha/\rho_n^2| < 1$. This case corresponds to high fluid velocities (large n). A straightforward calculation leads to conditions for the occurrence of supercritical and subcritical bifurcations:

Supercritical condition:

$$\Gamma < \mathscr{A} + \tfrac{10}{3}(n\pi)^2. \tag{35}$$

Subcritical condition:

$$\Gamma > \mathscr{A} + \tfrac{10}{3}(n\pi)^2. \tag{36}$$

A very large and hence a nonphysical value of n would be required to make the term $(n\pi)^2$ dominant in the above inequalities. Since with increasing fluid velocity ρ the effective stiffness of the pipe diminishes, the model strongly prefers supercritical bifurcation. However, since only the first mode is relevant from the physical aspect, the interest is in the case below.

[2] Eq. (5.62) of Païdoussis (1998).

Case 2: $|(n\pi)^2/\alpha|<1$. This case may be attributed to low fluid velocities (small n), and is therefore of physical and practical interest. The corresponding conditions are:

Supercritical condition:

$$\Gamma < \mathscr{A} + \tfrac{7}{4}(n\pi)^2. \tag{37}$$

Subcritical condition:

$$\Gamma > \mathscr{A} + \tfrac{7}{4}(n\pi)^2. \tag{38}$$

The inclusion of the curvature term in the model increases the effective axial flexibility (the term $\mathscr{A} + \frac{7}{4}(n\pi)^2$) as compared to the basic Thurman and Mote model (the term $A - \frac{1}{2}(n\pi)^2$), so a slight preference is given to the supercritical bifurcation. Taking into consideration that n is small, tensioning effects may be dominant, for example in the case of short pipes with high flexural rigidity.

3.3. The effect of elastic support

In order to investigate the essential features of the added elastic support which involves distributed springs along the length of the pipe, we use the least complex model, the model of Thurman and Mote model without gravity effects. The corresponding equations are

$$w'''' + (\rho^2 - \Gamma)w'' + (\Gamma - \mathscr{A})(w''u' + u'w'' + \tfrac{3}{2}w'^2w'') + Kw = 0, \tag{39}$$

$$(\rho^2 - \mathscr{A})u'' + (\Gamma - \mathscr{A})w'w'' = 0. \tag{40}$$

The algebraic bifurcation equation is identical to Eq. (26), hence elastic support does not change the form of the solution in the vicinity of bifurcation points. From the equation for the kernel of the corresponding linear operator,

$$L\begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} w'''' + (\rho^2 - \Gamma)w'' + Kw \\ u'' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and Eq. (24), the relation determining position of bifurcation points is obtained:

$$\rho_n = (n\pi)\left[1 + \frac{\Gamma}{(n\pi)^2} + \frac{K}{(n\pi)^4}\right]^{1/2} = n\pi + \frac{1}{2}\frac{\Gamma}{(n\pi)} + \frac{1}{2}\frac{K}{(n\pi)^3} + \vartheta(2), \quad n = 0, 1, 2, \ldots \,. \tag{41}$$

Clearly if the spring constant is small, then the nontrivial bifurcation points are located at same positions ρ_n as in the Thurman and Mote model. The effect of elastic support is important for very small n (e.g. $n = 1$ or $n = 2$) and large values of K. However, if K is large, and $(1/2)\Gamma/n\pi$ small in comparison with $(n\pi)$, their locations may be distributed as in Fig. 5. A brief analysis of the above expression shows that the first nontrivial bifurcation point and some of the subsequent ones (usually for the first few n's) are to a large degree determined by the value of K, and the same applies to the distances between the bifurcation points. Afterwards, the term $(n\pi)$ dominates and the bifurcation points are equally distributed. A complete nonlinear model consisting of Eqs. (19) and (20) and including the term corresponding to the elastic support is very interesting, both from mathematical and physical aspects; however, due to its complexity it will be analyzed elsewhere (Rajković and Nikolić, 2005).

4. The complete nonlinear model of Païdoussis

4.1. Symmetry considerations

The derivation of the complete nonlinear model, based on the work (Semler et al., 1994), in its correct form is given in Païdoussis (2003). This model, as presented in Section 2, in contrast to the previously analyzed models, includes the effects of gravity and pressure at the downstream end, so the dimensionless equations of motion of an extensible cylinder conveying fluid have the form:

$$\begin{aligned} &w'''' + (\rho^2 - (\Gamma - \Pi))w'' + \gamma w' + (\Gamma - \mathscr{A} - \Pi)(w''u' + u'w'' + \tfrac{3}{2}w'^2w'') \\ &\quad - (3u'''w'') + 4u''w''' + 2u'w'''' + w'u'''' + 2w''^3 + 2w'^2w'''' + 8w'w''w''') \\ &\quad - \gamma\left[w'u' + \tfrac{1}{2}w'^3 - (1-\xi)\left(-w'' + u''w' + u'w'' + \tfrac{3}{2}w'^2w''\right)\right] = 0, \end{aligned} \tag{42}$$

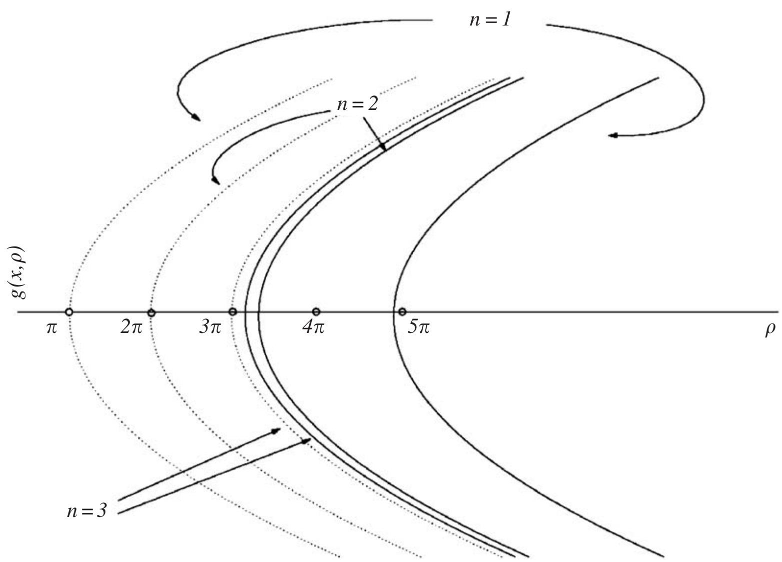

Fig. 5. Supercritical bifurcation for the fluid conveying pipe with and without elastic support. Dashed lines correspond to the case without elastic support. The case with elastic support assumes $K \geqslant 0$, and $(1/2)\Gamma/n\pi \ll n\pi$. The first three modes are shown.

$$(\rho^2 - \mathscr{A})u'' - (w''w''' + w'w'''') - \gamma\left[\tfrac{1}{2}w'^2 - (1-\xi)w'w''\right] + (\Gamma - \mathscr{A} - \Pi)w'w'' = 0. \tag{43}$$

The relevant quantities have been defined in the previous section.

To begin with, it is of interest to inspect the symmetry properties of this equation. It is immediately clear that in the first equation, only odd powers of w turn up, while this is not the case with the second equation. Hence, solution of the set (42) and (43), $v = (w\ u)^{\mathrm{T}}$, satisfies the following symmetry condition:

$$T\begin{pmatrix} w(\xi) \\ u(\xi) \end{pmatrix} = \begin{pmatrix} -w(\xi) \\ u(\xi) \end{pmatrix}, \tag{44}$$

where T represents the operator of the symmetry group. In the vicinity of bifurcating solutions, the solutions of the complete model have the form

$$\begin{pmatrix} w(\xi) \\ u(\xi) \end{pmatrix} = \mu(\rho)\begin{pmatrix} w_0(\xi) \\ 0 \end{pmatrix} + \vartheta(\mu^2), \tag{45}$$

where $w_0(x)$ represents the solution of the linear equation

$$w'''' + [(\rho^2 - (\Gamma - \Pi)) - \gamma(1-\xi)]w'' + \gamma w' = 0. \tag{46}$$

Acting with the symmetry operator T on Eq. (45) one obtains

$$\begin{aligned} T\begin{pmatrix} w(\xi) \\ u(\xi) \end{pmatrix} &= \mu(\rho)\begin{pmatrix} -w_0(\xi) \\ 0 \end{pmatrix} + \vartheta(\mu^2) \\ &= \mu(\rho)\begin{pmatrix} w_0(\xi) \\ 0 \end{pmatrix} + \vartheta(\mu^2). \end{aligned}$$

Hence, the bifurcation equation satisfies the relationship

$$g(-\mu, \rho) = -g(\mu, \rho),$$

so that g possesses the Z_2 symmetry. The above symmetry properties indicate that the bifurcation is necessarily of the pitchfork type (Golubitsky, 1985). The physical representation of this symmetry is a reflection across the longitudinal pipe axis. The use of only one equation in the model of Holmes, the one involving w, may be justified based on these symmetry considerations.

4.2. Critical velocity

With the fluid velocity as the bifurcation parameter, the procedure for obtaining critical velocity values corresponding to bifurcation points is almost the same as in the model of Holmes. Specifically, Eq. (46),

$$w'''' + [(\rho^2 - \Gamma + \Pi) - \gamma(1-\xi)]w'' + \gamma w' = w'''' + (\rho^2 - \Gamma + \Pi)w'' - \gamma\frac{\mathrm{d}}{\mathrm{d}\xi}[(1-\xi)w'] = 0, \tag{47}$$

corresponds to the linearized version of Eq. (8) of the Holmes' model. The only difference is the inclusion of the pressure term. The analogous equation to Eq. (B.8) in Appendix B determining bifurcation points is

$$\frac{2}{3}\frac{(\zeta^2+\gamma)^{3/2} - \zeta^3}{\gamma} = n\pi,$$

where

$$\zeta^2 = \rho^2 - (\Gamma - \Pi) - \gamma. \tag{48}$$

Taylor expanding and retaining terms to second order yields the following expression for the critical velocity values (bifurcation points):

$$\begin{aligned}\rho_n(L) &= n\pi + \frac{1}{4}\frac{\gamma}{n\pi} + \frac{1}{2}\frac{(\Gamma - \Pi)}{n\pi} + \vartheta(2) \\ &= n\pi + \frac{1}{4}\frac{\gamma + 2(\Gamma - \Pi)}{n\pi} + \vartheta(2), \quad n = 1, 2, \ldots \, .\end{aligned} \tag{49}$$

An important feature of the critical velocity ρ_n, in contrast to previously considered less complex models, is its dependence on the pipe length through the length dependence of Γ and Π. This dependence clearly diminishes with increasing velocity. Expression (49) is obtained assuming that fluid flow is in the direction of gravity ($\gamma > 0$). Bifurcations occur for velocity values higher than in the case when gravity, pressure and tension are not taken into account, under the assumption that

$$\gamma + 2\Gamma > 2\Pi. \tag{50}$$

The influence of gravity and tension is diminished by the pressure at the downstream end, as intuitively expected. The individual effect of the gravity term becomes dominant if the pressure value approaches the value of tension. If the flow is in the direction opposite to the direction of gravity ($\gamma < 0$), the expression for bifurcation points is

$$\begin{aligned}\rho_n(L) &= n\pi - \frac{1}{4}\frac{\gamma}{n\pi} + \frac{1}{2}\frac{(\gamma - \Pi)}{n\pi} + \vartheta(2) \\ &= n\pi - \frac{1}{4}\frac{\gamma - 2(\Gamma - \Pi)}{n\pi} + \vartheta(2), \quad n = 1, 2, \ldots \, .\end{aligned} \tag{51}$$

Clearly, gravity in this case acts in the same direction as pressure, and together they oppose the effects of tension. Additional insight into the position of bifurcation points, in units of velocity, may be obtained by considering expressions for interpoint distances. For $\gamma > 0$ the interpoint distance is

$$\begin{aligned}\Delta\rho_n(L) &= \pi - \frac{1}{4\pi}\frac{\gamma}{n(n+1)} - \frac{1}{2\pi}\frac{\Gamma - \Pi}{n(n+1)} \\ &= \pi - \frac{1}{4\pi n(n+1)}[\gamma - 2(\Gamma - \Pi)].\end{aligned} \tag{52}$$

As evident from the above expression, at low fluid velocities this distance is a sequence of increasing values provided that

$$\gamma = 2\Pi > 2\Gamma,$$

and a sequence of decreasing values for

$$\gamma + 2\Pi < 2\Gamma.$$

Since γ is length-dependent [Eq. (2)], the former condition is more easily satisfied for long pipes, while the latter is more likely fulfilled for short pipes. For large n (high velocities), the distance between bifurcation points has a fixed value of π.

For $\gamma < 0$, expression (52) becomes

$$\Delta\rho_n(L) = \pi + \frac{1}{4\pi}\frac{\gamma}{n(n+1)} - \frac{1}{2\pi}\frac{\Gamma - \Pi}{n(n+1)} = \pi + \frac{1}{4\pi n(n+1)}[\gamma - 2(\Gamma - \Pi)].$$

In this case, the distances between the bifurcation points form a decreasing sequence if

$$\gamma + 2\Pi > 2\Gamma,$$

and an increasing sequence provided that

$$\gamma + 2\Pi < 2\Gamma.$$

Recalling that $\gamma < 0$, the first condition is more likely in short pipes while the second one is more probable in long pipes.

4.3. Normal form of the bifurcation equation

The normal form of the pitchfork bifurcation modulo higher order terms (h.o.t.), reads

$$g(\mu, \rho) = g_{\mu\mu\mu}\mu^3 + g_{\mu\rho}\mu\rho + \text{h.o.t.} \tag{53}$$

In Appendix B, terms $g_{\mu\mu\mu}$ and $g_{\mu\rho}$ are defined with the symbol x replacing μ. In previously considered models the term $g_{\mu\rho}$ was explicitly determined and in all cases it was negative. Hence, the sign of $g_{\mu\mu\mu}$ determines whether the bifurcation is of supercritical or subcritical type. For the complete nonlinear model, the evaluation of terms $g_{\mu\mu\mu}$ and $g_{\mu\rho}$ is much more complicated, and in order to minimize computational effort it is sufficient to determine just the sign of $g_{\mu\rho}$. Following evaluation of the expression for $g_{\mu\rho}$ given in Appendix A, the following relationship is obtained:

$$g_{\mu\rho} = \langle v_0^*, L_\rho \cdot v_0 \rangle = 2\rho\langle v_0^*, v_0'' \rangle, \tag{54}$$

where L is the linear operator corresponding to the set of Eqs. (42) and (43)

$$L = \begin{pmatrix} w_0'''' + [(\rho^2 - (\Gamma - \Pi)) - \gamma(1-\xi)]w_0'' + \gamma w_0' \\ u_0'' \end{pmatrix}. \tag{55}$$

v_0 represents the solution of $L = 0$ evaluated using the procedure presented in Appendix B. With ζ given by expression (48), v_0 has the following form:

$$v_0 = \begin{pmatrix} w_0(\xi) \\ u_0(\xi) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(\zeta^2 + \gamma\xi)\left[\mathrm{J}_{2/3}\left(\frac{2}{3}\frac{\zeta^3}{\gamma}\right)\mathrm{J}_{-2/3}\left(\frac{2}{3}\frac{(\zeta^2+\gamma\xi)^{3/2}}{\gamma}\right) - \mathrm{J}_{-2/3}\left(\frac{2}{3}\frac{\zeta^3}{\gamma}\right)\mathrm{J}_{2/3}\left(\frac{2}{3}\frac{(\zeta^2+\gamma\xi)^{3/2}}{\gamma}\right)\right], \tag{56}$$

satisfying boundary conditions

$$v_0(0) = v_0(1) = 0.$$

Explicit expression of the inner product in (54) yields

$$g_{\mu\rho} = 2\rho\int_0^1 v_0 v_0''\,\mathrm{d}\xi.$$

Performing integration by parts and using boundary conditions, one obtains

$$g_{\mu\rho} = 2\rho[(v_0v_0')|_0^1 - \int_0^1 v_0^2\,\mathrm{d}\xi = -2\rho\int_0^1 v_0^2\,\mathrm{d}\xi.$$

Since the velocity $\rho > 0$ and

$$\int_0^1 v_0^2\,\mathrm{d}\xi > 0,$$

it follows that

$$g_{\mu\rho} = -2\rho \int_0^1 v_0^2 \, d\xi < 0. \tag{57}$$

Hence, once the sign of $g_{\mu\rho}$ is known, an explicit evaluation of this term is unnecessary. However, an explicit evaluation of the term $g_{\mu\mu\mu}$ is essential in order to obtain conditions determining the bifurcation type.

Details related to the evaluation of $g_{\mu\mu\mu}$, due to its mathematical complexity and extensiveness, are presented in Appendix C. Only results relevant for the final form of the expression determining the sign of $g_{\mu\mu\mu}$ are presented in this section. Following appropriate calculations, $g_{\mu\mu\mu}$ assumes the following form:

$$g_{\mu\mu\mu} = -\left[\Lambda(\zeta)\frac{(\Gamma - \mathscr{A} - \Pi + \gamma)}{\rho_n^2 - \mathscr{A}}\left(\frac{2}{3}\frac{\zeta^3}{\gamma}\right) + \Delta(\zeta)\frac{1}{\rho_n^2 - \mathscr{A}}\left(\frac{2}{3\gamma}\right)^{1/3}\zeta + \Omega(\zeta)\right]. \tag{58}$$

Terms $\Lambda(\zeta), \Delta(\zeta)$ and $\Omega(\zeta)$ are polynomial functions of ζ and their explicit representation may be found in Appendix C. As in previous, less complicated models, two cases may be considered.

Case 1: $\gamma/\zeta^2 < 1$. In the high velocity limit $\rho_n^2 - \mathscr{A} > 0$, so that the dominant part of (58) determining the sign of $g_{\mu\mu\mu}$ is

$$-2\Lambda(\zeta)(\Gamma - \mathscr{A} - \Pi + \gamma). \tag{59}$$

Hence, the bifurcation is supercritical or subcritical if the above expression is greater or less than zero, respectively. In order to be precise, it is informative to consider expression $\Lambda(\zeta)$. Explicitly, this expression is

$$\Lambda(\zeta) = \left[a\left(\frac{2}{3}\gamma\right)^2\left(\mathrm{J}_{-2/3}\left(\frac{2}{3}\frac{\zeta^3}{\gamma}\right)\right)^3 \mathrm{J}_{2/3}\left(\frac{2}{3}\frac{\zeta^3}{\gamma}\right) + b\left(\frac{2}{3}\gamma\right)^{8/3}\left(\mathrm{J}_{-2/3}\left(\frac{2}{3}\frac{\zeta^3}{\gamma}\right)\right)^4\right], \tag{60}$$

where a and b are the positive constants:

$$a = \frac{1}{(\Gamma(\frac{2}{3}))^2}\left(\frac{1}{\Gamma(\frac{1}{3})} + \frac{1}{\Gamma(\frac{1}{3})}\right)^2,$$

$$b = a\left(\frac{1}{\Gamma(\frac{1}{3})} + \frac{1}{\Gamma(\frac{1}{3})}\right)^{-1},$$

and $\Gamma(.)$ is the gamma function. The sign of $\Lambda(\zeta)$ is also positive as

$$b\left(\frac{2}{3}\gamma\right)^{8/3}\left(\mathrm{J}_{-2/3}\left(\frac{2}{3}\frac{\zeta^3}{\gamma}\right)\right)^4 > a\left(\frac{2}{3}\gamma\right)^2\left(\mathrm{J}_{-2/3}\left(\frac{2}{3}\frac{\zeta^3}{\gamma}\right)\right)^3 \mathrm{J}_{2/3}\left(\frac{2}{3}\frac{\zeta^3}{\gamma}\right),$$

irrespective of the possible negative signs of either $\mathrm{J}_{2/3}$ or $\mathrm{J}_{-2/3}$. Therefore, conditions for supercritical and subcritical bifurcations are determined by the term $-(\Gamma - \mathscr{A} - \Pi + \gamma)$, so that conditions are:

Supercritical condition:

$$\Gamma + \gamma < \mathscr{A} + \Pi, \tag{61}$$

Subcritical condition:

$$\Gamma + \gamma > \mathscr{A} + \Pi. \tag{62}$$

Recalling expressions for dimensionless system parameters (2), (3) and (21), inequalities (61) and (62) may be also expressed as

$$T(L) + (M + m)gL < EA + P(L), \tag{63}$$

and

$$T(L) + (M + m)gL > EA + P(L). \tag{64}$$

revealing their dependence on the length of the pipe. Physically, the effects of gravity are related to the length of the pipe [Eq. (2)], so larger γ may be associated with a longer pipe, and smaller γ with a shorter one. For short metal pipes, γ is rather small and its effects in inequalities (61) and (62) are weak. A comparison with corresponding inequalities for the Thurman and Mote model which includes curvature effects, shows that the pressure term replaces the velocity-dependent term. The reason is that the pressure term acts similarly as the velocity term, and it is clear that an adequately high level of pressurization alone may cause supercritical bifurcation. For pipes made of elastic material, gravity effects are important, as are the effects of axial flexibility, so that a supercritical bifurcation is more probable in shorter pipes. Intuitively, short pipes acquire a buckled shape in an evolutionary manner (corresponding to the supercritical case), while longer pipes suddenly deform (corresponding to the subcritical case). For long pipes, the tensioning term is small,[3] so that subcritical bifurcation is more likely to occur for low levels of pressurization. Thus, high pressurization in short pipes makes supercritical bifurcation more probable, while increasing the length of the pipe enhances the probability of subcritical bifurcation.

Case 2: $\zeta^2/\gamma<1$. Assuming that $\rho_n^2-\mathscr{A}<0$ in this low velocity case (which is of real physical importance) the sign-dominant term is

$$\varphi+(\Gamma-\mathscr{A}-\Pi+\gamma)\psi, \tag{65}$$

where φ and ψ are the following expressions:

$$\begin{aligned}\varphi = & -c_1\left(\frac{2}{3}\gamma\right)^{10/3}\left(\mathrm{J}_{-2/3}\left(\frac{2}{3}\frac{\zeta^3}{\gamma}\right)\right)^2\left(\mathrm{J}\left(\frac{2}{3}\frac{\zeta^3}{\gamma}\right)\right)^2-c_2\left(\frac{2}{3}\gamma\right)^{8/3}\left(\mathrm{J}_{-2/3}\left(\frac{2}{3}\frac{\zeta^3}{\gamma}\right)\right)^4\\ & -c_3\left(\frac{2}{3}\gamma\right)^{7/3}\left(\mathrm{J}_{-2/3}\left(\frac{2}{3}\frac{\zeta^3}{\gamma}\right)\right)^3\mathrm{J}_{2/3}\left(\frac{2}{3}\frac{\zeta^3}{\gamma}\right),\\ \psi = & \left(\mathrm{J}_{-2/3}\left(\frac{2}{3}\frac{\zeta^3}{\gamma}\right)\right)^4\left[\left(\frac{2}{3}\gamma\right)^{7/3}+\left(\frac{2}{3}\gamma\right)^2\frac{2^{1/3}}{(\Gamma(\frac{2}{3}))^2}\right]\left(\frac{1}{\Gamma(\frac{1}{3})}+\frac{1}{\Gamma(\frac{2}{3})}\right),\end{aligned} \tag{66}$$

and where c_1, c_2 and c_3 are constants (given in Appendix C). Clearly $\psi>0$ and it is easy to notice that the sign of $\varphi<0$ does not depend on signs of $\mathrm{J}_{-2/3}((2/3)\zeta^3/\gamma)$ and $\mathrm{J}_{2/3}((2/3)\zeta^3/\gamma)$, since the term containing these two Bessel functions is much smaller then the other two terms. Hence, the following conditions are obtained for bifurcation types:

Supercritical condition:

$$\Gamma+\gamma>\mathscr{A}+\Pi+\frac{|\varphi|}{\psi}, \tag{67}$$

Subcritical condition:

$$\Gamma+\gamma<\mathscr{A}+\Pi+\frac{|\varphi|}{\psi}. \tag{68}$$

Compared to conditions (61) and (62), inequalities pertaining to the low velocity case have an additional term acting in the same manner as the pressure and axial flexibility. The term $|\varphi|/\psi$ approximately proportional to γ so it counterbalances the gravity term on the left side of inequalities and the above inequalities may be further replaced with expressions

$$T(L)>EA+P(L),$$

and

$$T(L)<EA+P(L),$$

for subcritical and supercritical cases, respectively. Since tension is inversely proportional to the length of the pipe, for long pipes subcritical condition is practically always satisfied. For short pipes, supercritical bifurcation is possible only if the pressurization term is small compared to axial flexibility, since the corresponding condition may be written as

$$EA\left(\frac{1}{2L}\int_0^L w'^2\,\mathrm{d}\xi-1\right)>P(L). \tag{69}$$

In the above expression a viscoelastic material has been considered, and since we are considering a time independent model, axial tension is

$$T=\sigma A=(E\varepsilon+E^*\dot{\varepsilon})A=\varepsilon EA.$$

[3]Internal dissipation of the pipe material is assumed to be of the Kelvin–Voigt type.

where averaged axial strain ε due to lateral deflections w is

$$\varepsilon = \frac{1}{2L}\int_0^L w'^2 \,\mathrm{d}\xi.$$

The other possibility for this low velocity case, $\rho_n^2 - \mathscr{A} < 0$, has no physical meaning since n is small.

Although a subtle interplay between system parameters requires careful analysis under specific circumstances of interest, in general it may be concluded that in the case of high fluid velocity a supercritical bifurcation is more likely in a short pipe, while a subcritical bifurcation is more probable in a long pipe (provided that appropriate inequalities (61) and (62) are fulfilled). In the case of low fluid velocities (along with $\rho_n^2 > \mathscr{A}$) the situation is the opposite: a supercritical bifurcation is more likely in a long pipe, while subcritical bifurcation is preferred in short pipes provided condition (69) is satisfied.

5. Conclusion

Exact analytical solutions in the vicinity of the bifurcation point for each model are obtained, along with the derivation of conditions that classify bifurcations as supercritical or subcritical. Moreover, the analysis is performed in such a manner that the influence of important quantities on the dynamics of supported fluid-conveying pipes may be analyzed in the light of increasing complexity of each model.

Two important features of the stationary bifurcations for the supported fluid-conveying pipes should be emphasized. First, all bifurcations are of the pitchfork type as a consequence of reflection symmetry. Second, all perturbations of the pitchfork bifurcation, due to gravity or curvature effects for example, preserve the topological form of the unperturbed bifurcating diagram, due to the fact that 0 remains the solution of the perturbed equation. Consequently, unfolding of the bifurcation does not take place which requires that 0 is not the solution of the perturbed equation.

Models that consider both axial and lateral deflections, hence two-equation models, enable the possibility of both supercritical and subcritical pitchfork bifurcations. In contrast, the single equation model of Holmes which considers just transverse deflections allows only supercritical bifurcations. An important general characteristic of the classification of generic codimension-1 bifurcations into supercritical or subcritical is that the bifurcation type depends on only two factors: nonlinear terms and boundary conditions. Hence, nonlinear terms figuring in the equation for axial deflections make an important contribution to the terms defining the normal form of the bifurcation equation.

In the complete nonlinear model of Païdoussis, the critical velocity values at which bifurcation occurs depend on tension, gravity and pressure. For $\gamma > 0$, gravity and tensioning exert matching effects reflected in shifting bifurcation values in the positive direction, with the shift due to gravity being one-half the corresponding tensioning shift. The effects of pressurization oppose effects of gravity and tension. For $\gamma < 0$, gravity and pressurization act in the same direction, while tensioning exerts an opposing effect. The critical velocity values for each model are presented in Table 1. A summary of conditions classifying bifurcations as supercritical and subcritical is presented in Tables 2 and 3. An increase of complexity of the bifurcation type conditions may be traced, starting with the least complex model of Thurman and Mote and ending with the complete nonlinear model of Païdoussis. The velocity-dependent term appearing in inequalities corresponding to the Thurman and Mote model with curvature transforms into an analogous

Table 1
Critical velocity values for the models of Holmes, Thurman and Mote (TM), Thurman and Mote with curvature (TM-κ^2) and the complete nonlinear model of Païdoussis

Model	Critical velocity ρ_n	
	$\gamma \geqslant 0$	$\gamma < 0$
Holmes	$n\pi + \frac{1}{4}\frac{\gamma}{n\pi} + \frac{1}{2}\frac{\Gamma}{n\pi}$	$n\pi - \frac{1}{4}\frac{\gamma}{n\pi} + \frac{1}{2}\frac{\Gamma}{n\pi}$
TM and TM-κ^2 ($\gamma = 0$)	$n\pi + \frac{1}{2}\frac{\Gamma}{n\pi}$	—
TM-K ($\gamma = 0$)	$n\pi + \frac{1}{2}\frac{\Gamma}{(n\pi)} + \frac{1}{2}\frac{K}{(n\pi)^3}$	—
Complete nonlinear model	$n\pi + \frac{1}{4}\frac{\gamma}{n\pi} + \frac{1}{2}\frac{\Gamma}{n\pi} - \frac{1}{2}\frac{\Pi}{n\pi}$	$n\pi - \frac{1}{4}\frac{\gamma}{n\pi} + \frac{1}{2}\frac{\Gamma}{n\pi} - \frac{1}{2}\frac{\Pi}{n\pi}$

Table 2
Conditions for development of supercritical an subcritical bifurcations in the high velocity limit

Model	High velocity limit	
	Supercritical condition	Subcritical condition
TM	$\Gamma < \mathscr{A}$	$\Gamma > \mathscr{A}$
TM-κ^2	$\Gamma < \mathscr{A} + \frac{10}{3}(n\pi)^2$	$\Gamma > \mathscr{A} + \frac{10}{3}(n\pi)^2$
Complete nonlinear model	$\Gamma + \gamma < \mathscr{A} + \Pi$	$\Gamma + \gamma > \mathscr{A} + \Pi$

Table 3
Conditions for development of supercritical and subcritical bifurcations in different models of fluid conveying pipes. Term $|\varphi|/\psi$, proportional to γ, is defined in Eq. (66)

Model	Low velocity limit	
	Supercritical condition	Subcritical condition
TM	$\Gamma < \mathscr{A} - \frac{1}{2}(n\pi)^2$	$\Gamma > \mathscr{A} - \frac{1}{2}(n\pi)^2$
TM-κ^2	$\Gamma < \mathscr{A} + \frac{7}{4}(n\pi)^2$	$\Gamma > \mathscr{A} + \frac{7}{4}(n\pi)^2$
Complete nonlinear model	$\Gamma + \gamma > \mathscr{A} + \Pi + \frac{\|\varphi\|}{\psi}$	$\Gamma + \gamma < \mathscr{A} + \Pi + \frac{\|\varphi\|}{\psi}$

pressurizing effect in the most complex model. Setting $\gamma = \Pi = 0$, and assuming that Γ and Π are independent of the pipe length in the complete nonlinear model of Païdoussis, one obtains classification conditions for the Thurman and Mote model. Although conditions for determining whether the bifurcation is of supercritical or subcritical type involve a delicate interaction among system parameters (gravity effects, tensioning, pressurization and axial flexibility), it may be concluded that a general tendency is that for high fluid velocities supercritical bifurcation is more likely in short pipes, while a subcritical one is more likely in longer pipes (provided that appropriate unequalities (61) and (62) are satisfied). When the fluid velocity is low, but still high enough that its square exceeds axial flexibility, a supercritical bifurcation is more likely in longer pipes (with conditions (67) and (68) in effect), while a subcritical bifurcation may occur in short pipes provided that condition (69) is satisfied.

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Appendix A. Lyapunov–Schmidt reduction

Reduction of a nonlinear equation or a system of nonlinear equations

$$\Phi_i(y, \Lambda) = 0, \quad i = 1, \dots, n \tag{A.1}$$

to the single algebraic equation $g(x, \lambda)$ is the essential feature of the Lyapunov–Schmidt procedure. The vector $y = y(y_1, \dots, y_n)$ is the unknown in the above equation, while Λ is a vector of the parameters. We assume that only one parameter λ is of concern. The main starting assumption is

$$\Phi_i(0, 0) = 0, \tag{A.2}$$

and of interest is to describe the solutions of this system locally, in the vicinity of the origin. If the rank of the $n \times n$ Jacobian matrix $L = (d\Phi)_{0,0}$, is equal to n (nondegenerate case), the implicit function theorem guarantees the existence of solution y as a function of λ. If the rank is not equal to the size of the Jacobian matrix, we assume the minimally degenerate case, i.e.

$$\operatorname{rank} L = n - 1.$$

Assuming that $\Phi : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ is a smooth mapping, vector space complements M and N are chosen to $\ker L$ and $\operatorname{range} L$, respectively, so that

$$\mathbb{R}^n = \ker L \oplus M,$$

and

$$\mathbb{R}^n = N \oplus \operatorname{range} L,$$

where N denotes the null space of L. Introducing the projection operator $E : \mathbb{R}^n \to \operatorname{range} L$, the starting system of equation is expanded into

$$E\Phi(y, \lambda) = 0,$$

$$(I - E)\Phi(y, \lambda) = 0, \tag{A.3}$$

where $I - E$ is the complementary projection operator to E. Solving the first equation of (A.3) which fulfills the conditions of the implicit function theorem for $n - 1$ of the y variables, and inserting solutions in the second equation, yields an equation for the remaining one variable. Because of the splitting of $\mathbb{R}^n$, any vector $y \in \mathbb{R}^n$ may be written as $y = v(\in \ker L) + w(\in M)$, so that mapping

$$F : (\ker L) \times M \times \mathbb{R}^k \to \text{range } L,$$

is given by expression

$$F(v, w, \lambda) = E\Phi(v + w, \lambda).$$

The linear map

$$L : M \to \text{range } L$$

is invertible, thus according to the implicit function theorem it may be solved for w near the origin. Denoting this solution as $w = W(v, \lambda) : \ker L \times \mathbb{R}^k \to M$, which satisfies

$$E\Phi(v + W(v, \lambda), \lambda) = 0, \quad W(0, 0) = 0,$$

and it may be substituted into the second equation of (A.3) to obtain the reduced mapping $\phi : \ker L \times \mathbb{R}^k \to N$, where

$$\phi(v, \lambda) = (I - E)\Phi(v + W(v, \lambda), \lambda).$$

Consequently, the zeros of the $\phi(v, \lambda)$ are in one-to-one correspondence with the zeros of $\Phi(y, \lambda)$, or explicitly

$$\phi(v, \lambda) = 0 \quad \text{if and only if} \quad \Phi(v + W(v, \lambda), \lambda) = 0.$$

The reduced function $\phi(v, \lambda)$ may further be used to obtain the algebraic equation

$$g(x, \lambda) = \langle v_0^*, \phi(xv_0, \lambda) \rangle,$$

where $v_0^* \in (\operatorname{range} L)^\perp$, and $\langle ., . \rangle$ represents the standard inner product. Illustration of the reduction procedure is shown in Fig. A1. Finally, the derivatives of g figuring in the Taylor expansion of $g(x, \lambda)$ around the origin, after computation in terms of the original mapping $\Phi(y, \lambda)$, are given below:

$$g_x = 0,$$

$$g_{xx} = \langle v_0^*, \mathrm{d}^2\Phi(v_0, v_0) \rangle,$$

$$g_{xxx} = \langle v_0^*, \mathrm{d}^3\Phi(v_0, v_0, v_0) - 3\,\mathrm{d}^2\Phi(v_0, L^{-1}E\,\mathrm{d}^2\Phi(v_0, v_0)) \rangle,$$

$$g_\lambda = \langle v_0^*, \Phi_\lambda \rangle,$$

$$g_{\lambda x} = \langle v_0^*, \mathrm{d}\Phi_\lambda \cdot v_0 - \mathrm{d}^2\Phi(v_0, L^{-1}E\Phi_\lambda \rangle.$$

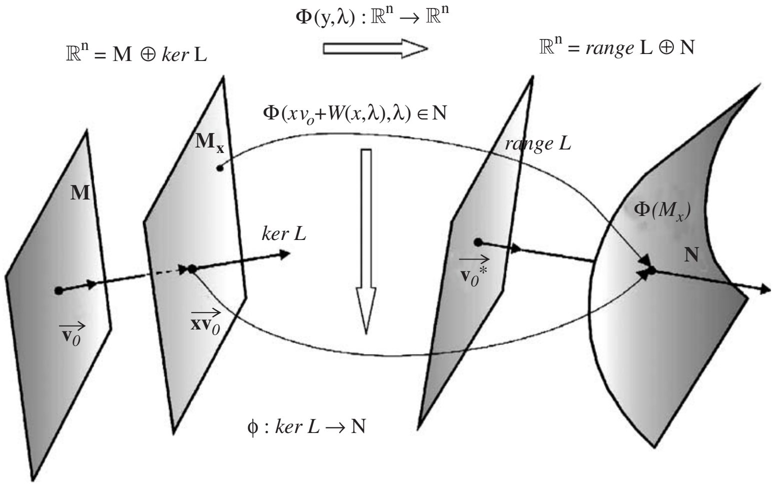

Fig. A1. A geometrical interpretation of the Lyapunov–Schmidt reduction.

Appendix B. Gravity effects in the model of Holmes

Solution of Eq. (11),

$$t^2 w'' + t w' + (t^2 - (\tfrac{1}{3})^2) w = 0, \tag{B.1}$$

may be expressed as

$$w(t) = c_1 \mathrm{J}_{1/3}(t) + c_2 \mathrm{J}_{-1/3}(t)$$

so that solution of Eq. (11) may be written as

$$u(t) = \frac{\sqrt{\alpha^2 + \gamma\xi}}{\gamma^{1/3}} \left[c_1 \mathrm{J}_{1/3}\left(\frac{2}{3} \frac{(\alpha^2 + \gamma\xi)^{3/2}}{\gamma} \right) + c_2 \mathrm{J}_{-1/3}\left(\frac{2}{3} \frac{(\alpha^2 + \gamma\xi)^{3/2}}{\gamma} \right) \right]. \tag{B.2}$$

Bessel functions possess the following well known properties, cf. Gradshteyn and Ryzhik (1994):

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{J_v(\dot{x})}{x^v} \right) = -\frac{\mathrm{J}_{v+1}(x)}{x^v},$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^v \mathrm{J}_v(x)) = x^v \mathrm{J}_{v-1}(x), \tag{B.3}$$

so that

$$\int x^{-v+1} \mathrm{J}_v(x)\,\mathrm{d}x = -x^{-v+1} \mathrm{J}_{v-1}(x),$$

$$\int x^{v+1} \mathrm{J}_v(x)\,\mathrm{d}x = x^{v+1} \mathrm{J}_{v-1}(x). \tag{B.4}$$

Furthermore, the boundary conditions of Eq. (11)

$$u'|_{\xi=0} = 0$$

$$u'|_{\xi=1} = 0,$$

yield

$$c_1 \mathrm{J}_{-2/3}\left(\frac{2}{3} \frac{\alpha^3}{\gamma} \right) - c_2 \mathrm{J}_{2/3}\left(\frac{2}{3} \frac{\alpha^3}{\gamma} \right) = 0,$$

$$c_1 J_{-2/3}\left(\frac{2}{3}\frac{(\alpha^2+\gamma)^{3/2}}{\gamma}\right) + c_2 J_{2/3}\left(\frac{2}{3}\frac{(\alpha^2+\gamma)^{3/2}}{\gamma}\right) = 0. \tag{B.5}$$

The condition for obtaining nontrivial solutions $c_1, c_2 \neq 0$ requires

$$J_{2/3}\left(\frac{2}{3}\frac{\alpha^3}{\gamma}\right) J_{-2/3}\left(\frac{2}{3}\frac{(\alpha^2+\gamma)^{3/2}}{\gamma}\right) + J_{2/3}\left(\frac{2}{3}\frac{(\alpha^2+\gamma)^{3/2}}{\gamma}\right) J_{-2/3}\left(\frac{2}{3}\frac{\alpha^3}{\gamma}\right) = 0. \tag{B.6}$$

Assuming weak gravitational influence ($\gamma \simeq 0$), the arguments of the Bessel functions in the above equation are large so that asymptotic expressions for Bessel functions may be used

$$J_v(x) = \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\pi}{2}v - \frac{\pi}{4}\right) + \vartheta(x^{-3/2}),$$

$$J_{-v}(x) = \sqrt{\frac{2}{\pi x}}\cos\left(x + \frac{\pi}{2}v - \frac{\pi}{4}\right) + \vartheta(x^{-3/2}).$$

Hence,

$$J_{2/3}(x) = \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{17\pi}{12}\right),$$

$$J_{-2/3}(x) = \sqrt{\frac{2}{\pi x}}\cos\left(x + \frac{17\pi}{12}\right). \tag{B.7}$$

In a straightforward manner, condition (B.6) yields

$$\frac{2}{3}\frac{(\alpha^2+\gamma)^{3/2} - \alpha^3}{\gamma} = n\pi. \tag{B.8}$$

Recalling that $\alpha^2 + \gamma = \rho^2$, the above condition is equivalent to

$$\rho^3 - (\rho^2 - \gamma)^{3/2} = \frac{3}{2}n\pi\gamma,$$

which, assuming $\gamma \ll \rho$ may be Taylor expanded, and retaining terms to second order yields

$$\rho_k = n\pi + \frac{\gamma}{4k\pi} + \vartheta(\gamma^2).$$

Keeping in mind the substitutions introduced in order to cast the equation in an analytically solvable form, the solution we seek is

$$v(\xi) = \int_0^{\xi} u(\xi)\,d\xi, \quad \text{with } u(0) = 0. \tag{B.9}$$

Inserting Eq. (70) in (B.3) the following expression is obtained:

$$v(x) = \frac{c_1}{\gamma^{1/3}}\left(\frac{3\gamma}{2}\right)^{2/3}\int_0^x \eta^{2/3} J_{1/3}(\eta)\,d\eta + \frac{c_2}{\gamma^{1/3}}\left(\frac{3\gamma}{2}\right)^{2/3}\int_0^x \eta^{2/3} J_{-1/3}(\eta)\,d\eta, \tag{B.10}$$

so that

$$v(\xi) = \tilde{c}_1\left[(\alpha^2+\gamma\xi) J_{-2/3}\left(\frac{2}{3}\frac{(\alpha^2+\gamma\xi)^{3/2}}{\gamma}\right) - \alpha^2 J_{-2/3}\left(\frac{2}{3}\frac{\alpha^3}{\gamma}\right)\right]$$
$$+ \tilde{c}_2\left[(\alpha^2+\gamma\xi) J_{2/3}\left(\frac{2}{3}\frac{(\alpha^2+\gamma\xi)^{3/2}}{\gamma}\right) - \alpha^2 J_{2/3}\left(\frac{2}{3}\frac{\alpha^3}{\gamma}\right)\right]. \tag{B.11}$$

Constants in the above two expressions satisfy the following relations:

$$\frac{c_2}{c_1}=\frac{\mathrm{J}_{-2/3}\left(\frac{2}{3}\frac{\alpha^3}{\gamma}\right)}{\mathrm{J}_{2/3}\left(\frac{2}{3}\frac{\alpha^3}{\gamma}\right)}=\frac{\tilde{c}_2}{\tilde{c}_1}. \tag{B.12}$$

Finally, the complete solution $v(x)$ may be written as

$$v(\xi)=\mu(\rho)(\alpha^2+\gamma\xi)\left[\mathrm{J}_{2/3}\left(\frac{2}{3}\frac{\alpha^3}{\gamma}\right)\mathrm{J}_{-2/3}\left(\frac{2}{3}\frac{(\alpha^2+\gamma\xi)^{3/2}}{\gamma}\right)-\mathrm{J}_{-2/3}\left(\frac{2}{3}\frac{\alpha^3}{\gamma}\right)\mathrm{J}_{2/3}\left(\frac{2}{3}\frac{(\alpha^2+\gamma\xi)^{3/2}}{\gamma}\right)+\vartheta(\xi^2)\right]; \tag{B.13}$$

for $\gamma\approx 0$, this acquires the asymptotic form

$$u(\xi)\simeq\frac{1}{(\alpha^2+\gamma\xi)^2}\sin\left[\frac{2}{3}\frac{(\alpha^2+\gamma\xi)^{3/2}-\alpha^3}{\gamma}\right], \tag{B.14}$$

while

$$\lim_{\gamma\to 0}u(xi)=\sin(n\pi\xi), \tag{B.15}$$

as expected.

Appendix C. Normal form of the bifurcation equation for the complete nonlinear model

The normal form of the pitchfork bifurcation modulo higher order terms (h.o.t), reads

$$g(\mu,\rho)=g_{\mu\mu\mu}\mu^3+g_{\mu\rho}\mu\rho+\text{h.o.t.}$$

As shown in the main part of the paper, explicit determination of the term $g_{\mu\rho}$ is not necessary since only its sign is relevant, and it was demonstrated that it is negative. The remaining term, $g_{\mu\mu\mu}$, requires explicit determination in order not only to evaluate its sign which enables classification of the bifurcation into supercritical or subcritical type, but also to extract conditions, in the form of inequalities, that need to be fulfilled in order for each bifurcation type to arise. In Appendix A, it was shown that this term requires evaluation of the inner product which may be written as

$$g_{xxx}=\langle v_0^*|\mathrm{d}^3\Phi(v_0,v_0,v_0)\rangle-\langle v_0^*|3\,\mathrm{d}^2\Phi(v_0,L^{-1}E\,\mathrm{d}^2\Phi(v_0,v_0))\rangle. \tag{C.1}$$

Noting that the adjoint operator L^* of

$$L=\begin{pmatrix} w_0''''+[(\rho^2-(\Gamma-\Pi))-\gamma(1-\xi)]w_0''+\gamma w_0' \\ u_0'' \end{pmatrix},$$

is equal to L, the following expression is obtained for the first inner product in (C.1):

$$\langle v_0^*|\mathrm{d}^3\Phi(v_0,v_0,v_0)\rangle=9\alpha_0\int_0^1 {v_0'}^2v_0''v_0\,\mathrm{d}\xi+9\gamma\int_0^1(1-\xi){v_0'}^2v_0''v_0\,\mathrm{d}\xi$$
$$-\left[3\gamma\int_0^1 {v_0'}^3v_0\,\mathrm{d}\xi+12\int_0^1 {v_0''}^3v_0\,\mathrm{d}\xi+12\int_0^1 {v_0'}^2v_0''''v_0\,\mathrm{d}\xi+48\int_0^1 v_0v_0'v_0''v_0'''\,\mathrm{d}\xi\right],$$

where

$$\alpha_0=\Gamma-\mathscr{A}-\Pi,$$

and where v_0 represents the solution of $L=0$ evaluated using procedure presented in Appendix B. With ζ given by expression (48) v_0 has the following form:

$$v_0=\begin{pmatrix} w_0(x) \\ u_0(x) \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \end{pmatrix}(\zeta^2+\gamma\xi)\left[\mathrm{J}_{2/3}\,\frac{2}{3}\frac{\zeta^3}{\gamma}\right)\mathrm{J}_{-2/3}\,\frac{2}{3}\frac{(\zeta^2+\gamma\xi)^{3/2}}{\gamma}\right)-\mathrm{J}_{-2/3}\,\frac{2}{3}\frac{\zeta^3}{\gamma}\right)\mathrm{J}_{2/3}\,\frac{2}{3}\frac{(\zeta^2+\gamma\xi)^{3/2}}{\gamma}\right)\right]. \tag{C.2}$$

Once the inverse operator $L^{-1}E\,\mathrm{d}^2\Phi(v_0,v_0)$ is evaluated following a lengthy procedure, the second inner product requires evaluation of the following integrals:

$$\begin{aligned}\langle v_0^*|3\,\mathrm{d}^2\Phi(v_0,L^{-1}E\,\mathrm{d}^2\Phi(v_0,v_0))\rangle &= (\alpha_0+\gamma)\int_0^1(\eta'\eta'')v_0v_0''\,\mathrm{d}\xi-\int_0^1\xi\eta' v_0v_0''\,\mathrm{d}\xi \\ &\quad -\int_0^1\xi\eta'' v_0v_0'\,\mathrm{d}\xi-3\int_0^1\eta''' v_0v_0''-4\int_0^1\eta'' v_0v_0'''\,\mathrm{d}\xi\,\mathrm{d}\xi \\ &\quad -2\int_0^1\eta' v_0v_0''''\,\mathrm{d}\xi-\int_0^1\eta'''' v_0'\,\mathrm{d}\xi-\gamma\int_0^1\eta' v_0v_0'\,\mathrm{d}\xi, \end{aligned} \tag{C.3}$$

where

$$\eta'=\frac{2(\alpha_0+\gamma)}{\rho^2-\mathscr{A}}\int v_0'v_0''\,\mathrm{d}\xi-\frac{2\gamma}{\rho^2-\mathscr{A}}\int\xi v_0'v_0''\,\mathrm{d}\xi-\frac{2}{\rho^2-\mathscr{A}}\int(v_0'v_0''''+v_0''v_0''')\,\mathrm{d}\xi-\frac{\gamma}{\rho^2-\mathscr{A}}\int {v_0'}^2\,\mathrm{d}\xi. \tag{C.4}$$

Evaluation of the integrals was performed using approximate expressions (B.7). Results were checked using lower and upper bounds for the Bessel functions, given by the following inequality (Neuman, 2004):

$$\frac{1}{\Gamma(\alpha+1)}\left(\frac{x}{2}\right)^{\alpha}\cos\left(\frac{x}{\sqrt{2(\alpha+1}}\right)\leqslant \mathrm{J}_\alpha\leqslant\frac{1}{\Gamma(\alpha+1)}\left(\frac{x}{2}\right)^{\alpha}\frac{1}{3(\alpha+1)}\left[2\alpha+1+(\alpha+2)\cos\left(\sqrt{\frac{3}{2\alpha+2}}x\right)\right]. \tag{C.5}$$

After lengthy calculations the following expression is obtained:

$$g_{\mu\mu\mu}=-\left[\Lambda(\zeta)\frac{(\Gamma-\mathscr{A}-\Pi+\gamma)}{\rho_n^2-\mathscr{A}}\left(\frac{2}{3}\frac{\zeta^3}{\gamma}\right)+\Delta(\zeta)\frac{1}{\rho_n^2-\mathscr{A}}\left(\frac{2}{3\gamma}\right)^{1/3}\zeta+\Omega(\zeta)\right], \tag{C.6}$$

where

$$\Lambda(\zeta)=\left[a\left(\frac{2}{3}\gamma\right)^2\left(\mathrm{J}_{-2/3}\left(\frac{2}{3}\frac{\zeta^3}{\gamma}\right)\right)^3\mathrm{J}_{2/3}\left(\frac{2}{3}\frac{\zeta^3}{\gamma}\right)+b\left(\frac{2}{3}\gamma\right)^{8/3}\left(\mathrm{J}_{-2/3}\left(\frac{2}{3}\frac{\zeta^3}{\gamma}\right)\right)^4\right]; \tag{C.7}$$

a and b are positive constants

$$a=\frac{1}{(\Gamma(\frac{2}{3}))^2}\left(\frac{1}{\Gamma(\frac{1}{3})}+\frac{1}{\Gamma(\frac{2}{3})}\right)^2$$

$$b=a\left(\frac{1}{\Gamma(\frac{1}{3})}+\frac{1}{\Gamma(\frac{2}{3})}\right)^{-1}.$$

Furthermore, term $\Delta(\zeta)$ is

$$\Delta(\zeta)=d_1\left(\frac{2}{3}\right)^{1/3}\gamma^{5/3}\left(\mathrm{J}_{-2/3}\left(\frac{2}{3}\frac{\zeta^3}{\gamma}\right)\right)^3\mathrm{J}_{2/3}\left(\frac{2}{3}\frac{\zeta^3}{\gamma}\right)+\left(\frac{2}{3}\gamma\right)^{10/3}\left(\mathrm{J}_{-2/3}\left(\frac{2}{3}\frac{\zeta^3}{\gamma}\right)\right)^4\frac{2^{1/3}}{(\Gamma(\frac{2}{3}))^2}\left(\frac{4}{39}\right), \tag{C.8}$$

where

$$d_1=\frac{1}{\Gamma(\frac{2}{3})}\left(\frac{1}{\Gamma(\frac{1}{3})}+\frac{1}{\Gamma(\frac{2}{3})}\right)\left(\frac{3}{\Gamma(\frac{1}{3})}+\frac{4}{\Gamma(\frac{2}{3})}\right).$$

Finally, term $\Omega(\zeta)$ is equal to

$$-(\varphi+(\Gamma-\mathscr{A}-\Pi+\gamma)\psi),$$

where

$$\varphi = -c_1\left(\frac{2}{3}\gamma\right)^{10/3}\left(\mathrm{J}_{-2/3}\left(\frac{2}{3}\frac{\zeta^3}{\gamma}\right)\right)^2\left(\mathrm{J}\left(\frac{2}{3}\frac{\zeta^3}{\gamma}\right)\right)^2 - c_2\left(\frac{2}{3}\gamma\right)^{8/3}\left(\mathrm{J}_{-2/3}\left(\frac{2}{3}\frac{\zeta^3}{\gamma}\right)\right)^4$$
$$- c_3\left(\frac{2}{3}\gamma\right)^{7/3}\left(\mathrm{J}_{-2/3}\left(\frac{2}{3}\frac{\zeta^3}{\gamma}\right)\right)^3\mathrm{J}_{2/3}\left(\frac{2}{3}\frac{\zeta^3}{\gamma}\right),$$
$$\psi = \left(\mathrm{J}_{-2/3}\left(\frac{2}{3}\frac{\zeta^3}{\gamma}\right)\right)^4\left[\left(\frac{2}{3}\gamma\right)^{7/3} + \left(\frac{2}{3}\gamma\right)^2\frac{2^{1/3}}{(\Gamma(\frac{2}{3}))^2}\right]\left(\frac{1}{\Gamma(\frac{1}{3})} + \frac{1}{\Gamma(\frac{2}{3})}\right);$$

c_1, c_2 and c_3 are the following constants:

$$c_1 = 12\frac{2^{1/3}}{\Gamma(\frac{4}{3})\Gamma(\frac{1}{3})}\left(\frac{1}{\Gamma(\frac{1}{3})} + \frac{1}{\Gamma(\frac{2}{3})}\right),$$

$$c_2 = \frac{1}{\Gamma(\frac{2}{3})}\left(\frac{1}{\Gamma(\frac{1}{3})} - \frac{1}{\Gamma(\frac{2}{3})}\right)^2,$$

$$c_3 = \frac{2^{1/3}}{\Gamma(\frac{2}{3})}\left(\frac{1}{\Gamma(\frac{4}{3})} - \frac{1}{\Gamma(\frac{1}{3})}\right).$$

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